

# The continuum limit of the Bell model

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(Dated: May 18, 2011)

## Abstract

In a paper entitled *Beables for Quantum Field Theory*, John Bell has shown that it was possible to build a realistic interpretation of any hamiltonian lattice quantum field theory involving Fermi fields. His model was constructed along the ideas he used to present the de Broglie-Bohm pilot wave theory. However, the beable (or element of reality) is now the fermion number density, which is not a particle density, as in the de Broglie-Bohm pilot wave theory. The model is stochastic but Bell thought that it would become deterministic in the continuum limit. We show that it is indeed the case, under an assumption about the physical state of the universe, which follows naturally from the Bell model. Moreover, the continuum model can be established directly. The assumption is that the universe is in a state obtained from the positronic sea (all positron states occupied) by creating a finite number of negative charges. The physical interpretation is the following: the negative charges are in motion in the positronic sea and their positions are the beables of the Bell model. The velocity laws we obtain for the motion of the negative charges are very similar to those given by Bohm and his co-workers for free relativistic fermions (first quantization). The Bell model is non-local (it is unavoidable); we show it explicitly in the simplest case. Under the previous assumption about the state of the universe, and for quantum field theories involving only Fermi fields, wave functions can be defined, and calculations can be performed as in non-relativistic quantum mechanics, since we stay in a sector of the Fock space with a fixed fermion number.

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## I. INTRODUCTION

At the time Bell wrote his paper [1], Bohm had already shown that it was possible to build a realistic interpretation of any bosonic quantum field theory [2]. To achieve that goal, Bohm took the field as the beable (or element of reality), however he was not able to do the same for fermions. The aim of Bell was then to show that it was also possible to build a realistic interpretation of any fermionic quantum field theory, along the pilot-wave ideas given by de Broglie and later by Bohm. Bell managed doing so but he took a really different beable: the fermion number density. It is quite different from the non-relativistic pilot-wave theory, whose beables are the positions of the particles. The model is also formulated on a spatial lattice (space is discrete but time remains continuous). His model is stochastic, but he suspected that the theory would become deterministic in the continuum limit. The crucial point is that the fermion number density is not a particle density. It is in fact related to the charge density.

To deal with fermions on a spatial lattice, we have used the Banks-Susskind-Kogut theory of staggered fermions [3]. It is the best way to eliminate the fermion doubling problem. Taking the continuum limit of the Bell model for a staggered lattice, we show that the theory becomes indeed deterministic in that limit. The velocity laws we obtain are very similar to those given by Bohm and co-workers for free relativistic fermions (first quantization). We have worked with a one-dimensional lattice to simplify the expressions but we think that it could be extended to the three-dimensional case without difficulty. This remark is justified since we show that the continuum model can also be established directly.

Finally, we will study in more details the fact that any purely fermionic quantum field theory can be brought to a particular sector of the Fock space, with a fixed fermion number, allowing us to define wave functions and to perform calculations in the same way as they are done in non-relativistic quantum mechanics. We will also study entanglement and non-locality in the simplest case (two quanta of the fermion number).

## II. THE BELL MODEL

### A. Ontology

Three-space continuum is replaced by a finite lattice, whose sites are labelled by an index

$$l = 1, 2, \dots, L .$$

The fermion number density is the operator

$$F(l) = \psi^\dagger(l)\psi(l) = \psi_1^\dagger(l)\psi_1(l) + \dots + \psi_4^\dagger(l)\psi_4(l) .$$

Since

$$[F(k), F(l)] = 0 \quad \forall k, l \in \{1, 2, \dots, L\} , \quad (1)$$

it is possible to define eigenstates of the fermion number density:

$$F(l)|n, q\rangle = f(l)|n, q\rangle ,$$

where  $q$  are eigenvalues of observables  $Q$  such that  $\{F(1), F(2), \dots, F(L), Q\}$  is a complete set of observables, and  $n$  is a fermion number density configuration ( $n = \{f(1), f(2), \dots, f(L)\}$ ). Eigenvalues  $f(l)$  belong to  $\{0, 1, 2, 3, 4\}$ .

Thus we can imagine that the universe is in a definite fermion number density configuration  $n(t)$  at each time  $t$ , and that a measurement of the fermion number density at time  $t$  would simply reveal the configuration  $n(t)$ . In Bell's words, the fermion number density is given the beable status.

The second element, in the description of the universe, is the pilot-state  $|\Psi(t)\rangle$ . Hence the universe, at time  $t$ , is completely described by the couple  $(|\Psi(t)\rangle, n(t))$ .

### B. Equations of motion

For the pilot-state, the Schrödinger equation is retained:

$$i \frac{d|\Psi(t)\rangle}{dt} = H|\Psi(t)\rangle .$$

An equation of motion for  $n(t)$  must be added (it is called the velocity-law in pilot-wave theories). Call  $P_m(t)$  the probability for the universe to be in configuration  $m$  at time  $t$ .

Then the velocity-law must be such that the relation

$$P_m(t) = \sum_q |\langle m, q | \Psi(t) \rangle|^2 \quad (2)$$

holds for any time  $t$ , in order to reproduce the predictions of orthodox quantum field theory. Since the configuration space is discrete, it is impossible to find a deterministic velocity-law. Instead jump-rates have to be defined. Call  $T_{nm}(t)$  the jump-rate for the transition  $m \rightarrow n$  at time  $t$ , for  $m \neq n$  ( $T_{mm}(t) = 0 \ \forall m$ ). In other words,  $T_{nm}(t)$  is the probability density (probability by unit of time) for the universe to jump in configuration  $n$ , knowing that the universe is in configuration  $m$  at time  $t$ . The probability for the universe to stay in configuration  $m$  at time  $t + dt$  is obtained by the normalization condition and is equal to

$$1 - \sum_n T_{nm}(t)dt, \quad (3)$$

for  $dt$  small enough. Eq. (2) is assumed to be true for an initial time  $t_0$ ; then the constraint on the stochastic velocity-law becomes

$$\frac{dP_m(t)}{dt} = \sum_q \frac{d}{dt} |\langle m, q | \Psi(t) \rangle|^2. \quad (4)$$

Let's calculate the first member:

$$P_m(t + dt) = \sum_n T_{mn}(t)P_n(t)dt + (1 - \sum_n T_{nm}(t))P_m(t)dt, \quad (5)$$

from which follows

$$\frac{dP_m(t)}{dt} = \lim_{dt \rightarrow 0} \frac{P_m(t + dt) - P_m(t)}{dt} = \sum_n (T_{mn}(t)P_n(t) - T_{nm}(t)P_m(t)). \quad (6)$$

Now let's calculate the second member of equation (4). With the help of the Schrödinger equation, we have

$$\frac{d|\langle m, q | \Psi(t) \rangle|^2}{dt} = \langle m, q | \Psi(t) \rangle \langle \Psi(t) | iH | m, q \rangle + \langle m, q | -iH | \Psi(t) \rangle \langle \Psi(t) | m, q \rangle \quad (7)$$

$$= 2\Re[\langle m, q | -iH | \Psi(t) \rangle \langle \Psi(t) | m, q \rangle] \quad (8)$$

$$= 2 \sum_{n,p} \Re[\langle \Psi(t) | m, q \rangle \langle m, q | -iH | n, p \rangle \langle n, p | \Psi(t) \rangle] . \quad (9)$$

Hence the constraint on the velocity law is

$$\sum_n (T_{mn}(t)P_n(t) - T_{nm}(t)P_m(t)) = 2 \sum_{n,p,q} \Re[\langle \Psi(t) | m, q \rangle \langle m, q | -iH | n, p \rangle \langle n, p | \Psi(t) \rangle], \quad (10)$$

or

$$T_{mn}(t)P_n(t) - T_{nm}(t)P_m(t) = 2 \sum_{p,q} \Re[\langle \Psi(t) | m, q \rangle \langle m, q | - iH | n, p \rangle \langle n, p | \Psi(t) \rangle] . \quad (11)$$

If one takes for the following definition for the jump-rates:

$$T_{mn}(t) = \frac{J_{mn}(t)}{P_n(t)} \quad \text{if } J_{mn} \geq 0 , \quad (12)$$

$$T_{mn}(t) = 0 \quad \text{otherwise} , \quad (13)$$

where

$$J_{mn}(t) = 2 \sum_{p,q} \Re[\langle \Psi(t) | m, q \rangle \langle m, q | - iH | n, p \rangle \langle n, p | \Psi(t) \rangle] , \quad (14)$$

then Eq. (4) is satisfied.

### C. Comments

We would like to make some remarks about the fermion number density. First, eigenstates of the fermion number density are also eigenstates of the charge density, which is

$$-e : \psi^\dagger(l)\psi(l) : ,$$

at least if we consider only electrons and positrons. The fermion number

$$F = \sum_{s=1}^{s=2} \int d^3\vec{p} [c_s^\dagger(\vec{p})c_s(\vec{p}) + d_s(\vec{p})d_s^\dagger(\vec{p})] ,$$

is not the particle number, which is

$$N = \sum_{s=1}^{s=2} \int d^3\vec{p} [c_s^\dagger(\vec{p})c_s(\vec{p}) + d_s^\dagger(\vec{p})d_s(\vec{p})] .$$

In fact the fermion number density does not commute with the particle number; it is possible to find well-behaved functions  $f(\vec{x})$  such that

$$[ \int d^3\vec{x} f(\vec{x}) \psi^\dagger(\vec{x}) \psi(\vec{x}), N ] \neq 0 .$$

The proof is given in appendix A.

Thus the charge density does not commute with the particle number either. A measurement of the charge contained in any finite region including the coordinate  $\vec{x}_0$ , with value  $-e$ ,

is never an electron. In fact, as we will see, it is a superposition containing one electron, two electrons and one positron, three electrons and two positrons, and so on. On one hand, it is disturbing, since the tracks observed in bubble chambers are said to represent electrons or positrons. On the other hand, when the electromagnetic field is taken into account, it is quite natural, since measurement of localized properties involve high energy radiation, and thus that can lead to pair creations. But these are just few remarks to draw attention to the interpretation problems one has to cope with.

### III. THE DIRAC THEORY IN A 1 + 1 SPACE-TIME

#### A. The Dirac equation

##### 1. Solutions of the Dirac equation

The hamiltonian is

$$H = \alpha p + \beta m . \quad (15)$$

Since  $H^\dagger = H$  and  $H^2 = p^2 + m^2$ , we obtain the following relations:

$$\alpha^\dagger = \alpha \quad \beta^\dagger = \beta \quad \{\alpha, \beta\} = 0 \quad \alpha^2 = \beta^2 = 1 \quad . \quad (16)$$

The smallest dimension for a representation of that algebra is two. For example:

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad . \quad (17)$$

Thus we have a spinor of dimension 2,  $\psi(t, x)$ , which is solution of

$$i \frac{\partial \psi(t, x)}{\partial t} = -i \alpha \frac{\partial \psi(t, x)}{\partial x} + m \beta \psi(t, x) . \quad (18)$$

This equation can be rewritten in a covariant form, by introducing the  $\gamma$  matrices, defined by

$$\gamma^0 = \beta = \gamma_0 \quad \gamma^1 = \alpha = -\gamma_1 \quad . \quad (19)$$

That implies

$$(i \gamma^\mu \partial_\mu - m) \psi(t, x) = 0 . \quad (20)$$

One can verify that the  $\gamma$  matrices satisfy the following relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad \gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 \quad . \quad (21)$$

Each component of the spinor is a solution of the Klein-Gordon equation:

$$(\square + m^2)\psi(t, x) = 0 \quad . \quad (22)$$

Now we search for free solutions; the most general form is thus

$$u(p)e^{-iE_p t} e^{ipx} \quad v(p)e^{iE_p t} e^{-ipx} \quad , \quad (23)$$

with  $p \in \Re$  and  $E_p = \sqrt{p^2 + m^2}$ . We find that

$$\psi_+^p(t, x) = u(p)e^{-iE_p t} e^{ipx} = \begin{pmatrix} 1 \\ \frac{p}{m+E_p} \end{pmatrix} e^{-iE_p t} e^{ipx} \quad (24)$$

$$\psi_-^p(t, x) = u(p)e^{iE_p t} e^{-ipx} = \begin{pmatrix} \frac{p}{m+E_p} \\ 1 \end{pmatrix} e^{iE_p t} e^{-ipx} \quad (25)$$

are solutions of Dirac equation, respectively of positive and negative energy. To obtain an interpretation of the theory, we need a conserved current, whose temporal component is positive:

$$\partial_\mu j^\mu = 0 \quad \text{with } j^0 \geq 0 \quad . \quad (26)$$

The current

$$j^\mu = \bar{\psi} \gamma^\mu \psi \quad , \quad (27)$$

with  $\bar{\psi} = \psi^\dagger \gamma^0$ , is suitable. Spinors normalization still remains to be discussed. Assume that the universe is a box of volume  $V$ , in a inertial frame  $\Sigma$  where the momentum of the free particle is  $p$ . Then the quantity

$$\int_V dx j^0(t, x)$$

must be equal to 1 in every inertial frame. That means that the spinors  $u(p)$  and  $v(p)$  must be normalized to  $\frac{1}{V_0} \frac{E_p}{m}$ , where  $V_0$  is the volume of the universe in an inertial frame where the particle is at rest.

## 2. Physical interpretation

Dirac obtained a conserved current, whose time-component is positive, a task that was impossible with the Klein-Gordon equation. But the negative energy states are still there. Once interactions are taken into account, that would lead to the instability of the hydrogen atom, for example. To avoid this, Dirac assumed that all negative energy states were occupied. Hence a positive energy electron cannot transit to a negative energy state, due to Pauli exclusion principle. That state of lowest energy is called the Dirac sea. It is impossible to distinguish it from a state where no electrons are present. The absence of a negative energy state of momentum  $p$  ( a hole in the Dirac sea) would be seen as a particle of positive energy  $\sqrt{p^2 + m^2}$ , momentum  $-p$  and charge  $e$ . That led to the prediction of anti-particles known as positrons.

### B. The Dirac quantum field theory

The first step, in the construction of the corresponding quantum field theory, is to obtain a classical relativistic action, from which we can obtain the Dirac equation, by using the least action principle. The following action is suitable:

$$S[\psi, \psi^\dagger] = \int dx dt \bar{\psi}(t, x) (i\gamma^\mu \partial_\mu - m) \psi(t, x) . \quad (28)$$

It is not hermitian but it can be rewritten as

$$S = S_h + \int \partial_\mu J^\mu ,$$

where  $S_h$  is hermitian, and the last term can be dropped. We get the momenta conjugate to the fields:

$$\pi_a(t, x) = \frac{\partial \mathcal{L}(t, x)}{\partial(\dot{\psi}_a(t, x))} = i\psi_a^*(t, x) \quad \pi_a^*(t, x) = \frac{\partial \mathcal{L}(t, x)}{\partial(\dot{\psi}_a^*(t, x))} = 0 . \quad (29)$$

The next step is quantization, according to the canonical equal-time anti-commutation relations: classical fields become quantum fields, obeying the relation

$$\{\psi_a(t, x), \pi_b(t, y)\} = i\delta(x - y)\delta_{ab} \quad \{\psi_a(t, x), \psi_b(t, y)\} = 0 \quad \{\pi_a(t, x), \pi_b(t, y)\} = 0 .$$

Those relations can be rewritten as

$$\{\psi_a(t, x), \psi_b^\dagger(t, y)\} = \delta(x - y)\delta_{ab} \quad \{\psi_a(t, x), \psi_b(t, y)\} = 0 . \quad (30)$$



Since the quantum field  $\psi(t, x)$  satisfies the Dirac equation, it is a superposition of free solutions with operators as coefficients:

$$\psi(t, x) = \frac{1}{\sqrt{2\pi}} \int dp [c(p)u(p)e^{-iE_p t} e^{ipx} + \zeta(-p)v(p)e^{iE_p t} e^{-ipx}] , \quad (31)$$

$$\psi^\dagger(t, x) = \frac{1}{\sqrt{2\pi}} \int dp [c^\dagger(p)u^T(p)e^{iE_p t} e^{-ipx} + \zeta^\dagger(-p)v^T(p)e^{-iE_p t} e^{ipx}] . \quad (32)$$

$c$ ,  $\zeta$ ,  $c^\dagger$  and  $\zeta^\dagger$  are operators satisfying unknown anti-commutation relations, that must be chosen in order to regain the equal-time anti-commutation relations (Eq. (30)). Spinors are normalized to

$$u^\dagger(p)u(p) = \frac{E_p}{m} \quad v^\dagger(p)v(p) = \frac{E_p}{m} .$$

With the help of Eq. (31) and Eq. (32), we can work out the equal-time anti-commutation relations. We have

$$\begin{aligned} \{\psi_a(t, x), \psi_b(t, y)\} &= \frac{1}{2\pi} \int dp dq [\{c(p), c(q)\} e^{-iE_p t} e^{-iE_q t} e^{ipx} e^{iqy} u_a(p) u_b(q) \\ &\quad + \{\zeta(-p), \zeta(-q)\} e^{iE_p t} e^{iE_q t} e^{-ipx} e^{-iqy} v_a(p) v_b(q) \\ &\quad + \{c(p), \zeta(-q)\} e^{-iE_p t} e^{iE_q t} e^{ipx} e^{-iqy} u_a(p) v_b(q) \\ &\quad + \{\zeta(-p), c(q)\} e^{iE_p t} e^{-iE_q t} e^{-ipx} e^{iqy} v_a(p) u_b(q)] . \end{aligned}$$

If we take

$$\{c(p), c(q)\} = 0 \quad \{c(p), \zeta(q)\} = 0 \quad \{\zeta(p), \zeta(q)\} = 0 \quad \forall p, q , \quad (33)$$

then we obtain  $\{\psi_a(t, x), \psi_b(t, y)\} = 0$ . The relation  $\{\psi_a(t, x), \psi_b^\dagger(t, y)\} = \delta(x-y)\delta_{ab}$  remains to be considered. With the help of Eqs (31) and (32), we have

$$\begin{aligned} \{\psi_a(t, x), \psi_b^\dagger(t, y)\} &= \frac{1}{2\pi} \int dp dq [\{c(p), c^\dagger(q)\} e^{-iE_p t} e^{iE_q t} e^{ipx} e^{-iqy} u_a(p) u_b(q) \\ &\quad + \{\zeta(-p), \zeta^\dagger(-q)\} e^{iE_p t} e^{-iE_q t} e^{-ipx} e^{iqy} v_a(p) v_b(q) \\ &\quad + \{c(p), \zeta^\dagger(-q)\} e^{-iE_p t} e^{-iE_q t} e^{ipx} e^{iqy} u_a(p) v_b(q) \\ &\quad + \{\zeta(-p), c^\dagger(q)\} e^{iE_p t} e^{iE_q t} e^{-ipx} e^{-iqy} v_a(p) u_b(q)] . \end{aligned}$$

Taking

$$\{c(p), c^\dagger(q)\} = \frac{m}{E_p} \delta(p-q) \quad \{\zeta(p), \zeta^\dagger(q)\} = \frac{m}{E_p} \delta(p-q) \quad \{c(p), \zeta^\dagger(q)\} = 0 \quad \forall p, q ,$$

the relations (30) are regained. We can also choose

$$\psi(t, x) = \frac{1}{\sqrt{2\pi}} \int dp \sqrt{\frac{m}{E_p}} [c(p)u(p)e^{-iEt}e^{ipx} + \zeta(-p)v(p)e^{iEt}e^{-ipx}] , \quad (34)$$

with the following anti-commutation relations:

$$\begin{aligned} \{c(p), c(q)\} &= 0 & \{c(p), \zeta(q)\} &= 0 & \{\zeta(p), \zeta(q)\} &= 0 \\ \{c(p), c^\dagger(q)\} &= \delta(p - q) & \{\zeta(p), \zeta^\dagger(q)\} &= \delta(p - q) & \{c(p), \zeta^\dagger(q)\} &= 0 \quad \forall p, q . \end{aligned}$$

That is the choice we adopt. Now the observables can be expressed in the momentum space.

For the hamiltonian, we have

$$H = \int dx \psi^\dagger(x) [-i\alpha \nabla + m\beta] \psi(x) = \int dp \sqrt{p^2 + m^2} [c^\dagger(p)c(p) - \zeta^\dagger(-p)\zeta(-p)] .$$

The momentum is

$$P = \int dx \psi^\dagger(x) [-i\nabla] \psi(x) = \int dp p [c^\dagger(p)c(p) - \zeta^\dagger(-p)\zeta(-p)] .$$

And the fermion number is

$$F = \int dx \psi^\dagger(x) \psi(x) = \int dp [c^\dagger(p)c(p) + \zeta^\dagger(-p)\zeta(-p)] .$$

We can define a vacuum as a state annihilated by any operator  $c(p)$  or  $\zeta(p)$ ; we call that state  $|0_1\rangle$ :

$$c(p)|0_1\rangle = 0 \quad \zeta(p)|0_1\rangle = 0 \quad \forall p .$$

Now it is clear that  $c^\dagger(p)$  creates an electron of energy  $\sqrt{p^2 + m^2}$  and momentum  $p$ , whereas  $\zeta^\dagger(p)$  creates an electron of energy  $-\sqrt{p^2 + m^2}$  and momentum  $p$ , and that the Dirac sea is the state  $|DS\rangle = \prod_p \zeta^\dagger(p)|0_1\rangle$ . Usually, everything is rewritten by introducing positrons, by making the substitutions

$$\zeta^\dagger(p) \rightarrow d(-p) \quad \zeta(p) \rightarrow d^\dagger(-p) ,$$

where  $d^\dagger(p)$  is the operator that creates a positron of momentum  $p$  and energy  $\sqrt{p^2 + m^2}$ .

And we have to define another vacuum,  $|0_2\rangle$ :

$$c(p)|0_2\rangle = 0 \quad d(p)|0_2\rangle = 0 \quad \forall p .$$

We have the relations

$$|0_2\rangle = \prod_p \zeta^\dagger(p)|0_1\rangle \quad |0_1\rangle = \prod_p d^\dagger(p)|0_2\rangle .$$

#### IV. FERMIONS ON A LATTICE

One-space continuum is replaced by a lattice of spacing  $\delta$ , having  $L = 2N$  sites. The momentum space is also a lattice, having  $2N$  sites and a spacing  $\frac{\pi}{N\delta}$ . To get the lattice action, one makes the following substitutions

$$\int dx \rightarrow \delta \sum_l \quad \psi(t, x) \rightarrow \psi(t, j) \quad \frac{\partial \psi(t, x)}{\partial x} \rightarrow \frac{\psi(t, j+1) - \psi(t, j-1)}{2\delta}$$

in the continuum action

$$S[\psi, \psi^\dagger] = \int dx dt \bar{\psi}(t, x) [i\gamma^\mu \partial_\mu - m] \psi(t, x) .$$

Doing so, we obtain

$$S_{lat}[\psi, \psi^\dagger] = \delta \sum_j \int dt [i\psi^\dagger(j) \partial_t \psi(j) + i\psi^\dagger(j) \alpha \frac{\psi(t, j+1) - \psi(t, j-1)}{2\delta} - m\psi^\dagger(j) \beta \psi(j)] .$$

We can eliminate the factor  $\delta$ , by making the substitution

$$\psi(t, j) \rightarrow \frac{\psi(t, j)}{\sqrt{\delta}} ,$$

so that the lattice Dirac action is

$$S_{lat}[\psi, \psi^\dagger] = \sum_j \int dt [i\psi^\dagger(j) \partial_t \psi(j) + i\psi^\dagger(j) \alpha \frac{\psi(t, j+1) - \psi(t, j-1)}{2\delta} - m\psi^\dagger(j) \beta \psi(j)] .$$

The least action principle gives the lattice Dirac equation (we use periodic conditions on the boundaries  $\psi_{-N} = \psi_N$ ):

$$i \frac{\partial \psi(t, j)}{\partial t} = -i\alpha \frac{\psi(t, j+1) - \psi(t, j-1)}{2\delta} + m\beta \psi(t, j) ,$$

whose free solutions are

$$\psi_+^p(t, j) = u(p_{lat}) e^{-iE_{lat}(p)t} e^{ipj\delta} = \left( \frac{1}{\frac{p_{lat}}{m+E_{lat}(p)}} \right) e^{-iE_{lat}(p)t} e^{ipj\delta} \quad (35)$$

$$\psi_-^p(t, j) = u(p_{lat}) e^{iE_{lat}t} e^{-ipj\delta} = \left( \frac{\frac{p_{lat}}{m+E_{lat}(p)}}{1} \right) e^{iE_{lat}(p)t} e^{-ipj\delta} , \quad (36)$$

where  $p_{lat} = \frac{\sin(p\delta)}{\delta}$  and  $E_{lat}(p) = \sqrt{p_{lat}^2 + m^2}$ . Now we turn to the quantization. The anti-commutation relations become

$$\{\psi_a(t, j), \psi_b^\dagger(t, k)\} = \delta_b^a \delta_k^j ,$$

and all other anti-commutators vanishing. Since it satisfies the lattice Dirac equation, the quantum field  $\psi(t, j)$  is a superposition of free solutions, with operators as coefficients:

$$\psi(t, j) = \sum_p \omega(p) [c(p)u(p_{lat})e^{-iE_{lat}(p)t}e^{ipj\delta} + d^\dagger(p)v(p_{lat})e^{iE_{lat}(p)t}e^{-ipj\delta}] .$$

Again,  $\omega(p)$  and the anti-commutation relations satisfied by the operators  $c$ ,  $d$ ,  $c^\dagger$  and  $d^\dagger$ , are determined by the canonical equal-time anti-commutation relations. But is not difficult to see that the hamiltonian is

$$H = \sum_p \sqrt{\frac{\sin^2(p\delta)}{\delta^2} + m^2} [c^\dagger(p)c(p) - d(p)d^\dagger(p)]$$

Thus there are four states of energy  $m$ . Generally, every eigenstate containing  $n$  particles is degenerate, with degeneracy  $2^n$ . The same problem occur in the propagator; it has four poles and thus propagates twice more particles. It is called the fermion doubling problem and there are many theories to deal with it (the Wilson theory, the Banks-Susskind-Kogut theory of staggered fermions, to mention the main ones). In the continuum limit (lattice spacing going to zero and finite momentum), we have

$$\frac{\sin(p\delta)}{\delta} \rightarrow p ,$$

so that the fermion doubling problem disappears. But that is not a reason to ignore it in our calculations. In the one-dimensional case, it is easier to use the Banks-Susskind-Kogut theory of staggered fermions to overcome it.

### A. The Banks-Susskind-Kogut theory of staggered fermions

The idea is to start from the previous theory, with a lattice spacing equal to  $2\delta$ , and to say that there are two superposed lattices, one where upper components of  $\psi(t, j)$  live, and another one where lower components live. By moving the lower lattice to the right, with a translation of magnitude  $\delta$ , we obtain a theory of a complex field  $\phi(l)$ , over a lattice containing twice more sites.

Since the part of the article which we are interested in is quite small, we will just quote it [3]:

Consider a spatial lattice (continuous time) with a lattice spacing  $a$ . Label the lattice sites with an integer  $n$ . There will be a one-component fermion field  $\phi(n)$

at each site  $n$ .  $\phi(n)$  satisfies the anti-commutation relation

$$\{\phi^\dagger(n), \phi(m)\} = \delta_{nm} \quad \{\phi(n), \phi(m)\} = 0 \quad (37)$$

$\phi(n)$  is related to a properly normalized continuum field  $\chi$  having canonical anti-commutation relations by

$$\phi(n) = \sqrt{a}\chi(x) \quad (38)$$

Consider the hamiltonian

$$H = \frac{i}{2a} \sum_n [\phi^\dagger(n)\phi(n+1) - \phi^\dagger(n+1)\phi(n)] \quad (39)$$

We claim that with a proper identification of a two-component fermion fields Eqs (37)-(39) generate the massless Dirac equation in the continuum limit. First compute

$$i[H, \phi(n)] = \dot{\phi}(n) = \frac{\phi(n+1) - \phi(n-1)}{2a} \quad (40)$$

Note that the time dependence of  $\phi(n)$  at even (odd) sites is determined by the spatial difference of  $\phi(n \pm 1)$  at odd (even) sites. So, to ensure finite time dependence in  $\phi(n)$  at even (odd) sites, we must require that the spatial dependence in  $\phi(n)$  at odd (even) sites be smooth. Thus we defined a two-component field  $\psi(n)$  as follows:

$$\psi = \begin{pmatrix} \psi_e \\ \psi_o \end{pmatrix}$$

$$\psi_e(n) = \phi(n), \quad n \text{ even} \quad (41)$$

$$\psi_o(n) = \phi(n), \quad n \text{ odd.}$$

Then the components of  $\psi(n)$  satisfy the equations

$$\dot{\psi}_o = \frac{\Delta\psi_e}{\Delta x}, \quad \dot{\psi}_e = \frac{\Delta\psi_o}{\Delta x} \quad (42)$$

where  $\Delta$  indicates the discrete difference in Eq. (40). Note that Eq. (42) becomes the massless Dirac equation in the continuum limit,

$$\frac{\partial}{\partial t}\psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\psi \quad (43)$$

in a standard basis where

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

If we consider the case of massive fermions, we just add a term  $m \sum_n (-1)^n \phi^\dagger(n) \phi(n)$  to the massless hamiltonian:

$$H = -\frac{i}{2\delta} \sum_n [\phi^\dagger(n) \phi(n+1) - \phi^\dagger(n+1) \phi(n)] + \sum_n m (-1)^n \phi^\dagger(n) \phi(n) .$$

There is a minus sign in the kinetic term, compared to the expression in the Banks-Susskind-Kogut article, but that is just a matter of conventions. Again we take the number of lattice sites to be  $2N$ . Thus the momentum space is a lattice containing  $2N$  sites, with spacing  $\frac{\pi}{N\delta}$ . What about the fermion doubling problem? The momentum lattice is still  $\frac{\sin(p\delta)}{\delta}$ , but the momentum space is divided into two sub-spaces:

- the sub-space  $\mathcal{P}_1$ , where electrons live.
- the sub-space  $\mathcal{P}_2$ , where positrons live.

And it is impossible to find  $p, q \in \mathcal{P}_1$  (resp.  $\mathcal{P}_2$ ) such that  $\frac{\sin(p\delta)}{\delta} = \frac{\sin(q\delta)}{\delta}$ . It is quite straightforward to show if we think in terms of degrees of freedom. Thus there are  $N$  sites in  $\mathcal{P}_1$  and  $N$  sites in  $\mathcal{P}_2$ .

### 1. *Eigenstates of the fermion number density*

The fermion number density is the following operator:

$$\phi^\dagger(l) \phi(l) .$$

The fermion number density is an operator with positive eigenvalues. Summing over all sites, we obtain the fermion number:

$$\sum_l \phi^\dagger(l) \phi(l) = \sum_{p \in \mathcal{P}_1} c^\dagger(p) c(p) + \sum_{p \in \mathcal{P}_2} d(p) d^\dagger(p) .$$

The eigenvalues of the fermion number range from 0 to  $2N$ . The state with the lowest fermion number is thus the positronic sea (all positron states occupied):

$$|0_1\rangle = \prod_{p \in \mathcal{P}_2} d^\dagger(p) |0_2\rangle .$$

Since the eigenvalues of the fermion number density are positive, there can be only one eigenstate of the fermion number density with fermion number equal to 0; it is thus the positronic sea:

$$\phi^\dagger(l)\phi(l)|0_1\rangle = 0 .$$

From the anti-commutation relation

$$\{\phi(l), \phi^\dagger(k)\} = \delta_{lk} ,$$

we find that  $\phi^\dagger(k)$  creates a quantum of the fermion number at site  $k$ , whereas  $\phi(k)$  destroys it. The eigenvalues of  $\phi^\dagger(l)\phi(l)$  belong to  $\{0, 1\}$ . The positronic sea is annihilated by any annihilator  $\phi(k)$  (it could be called the vacuum of the fermion number density). Eigenstates of the fermion number density are thus generated by applying creators at different sites.

## V. THE CONTINUUM LIMIT

The fermion number commutes with the hamiltonian

$$[H, \sum_l \phi^\dagger(l)\phi(l)] = 0 .$$

and any physical state is an eigenstate of the fermion number, so there is an integer  $\omega$  such that

$$\sum_l \phi^\dagger(l)\phi(l)|\Psi(t)\rangle = \omega|\Psi(t)\rangle .$$

for any time  $t$  (with  $\omega \in \{0, 1, \dots, 2N\}$ ).

### A. A localized quantum

Let's first consider the case of a localized quantum

$$\sum_l \phi^\dagger(l)\phi(l)|\Psi(t)\rangle = |\Psi(t)\rangle .$$

Since we are restricted to the sector of the Fock space with the fermion number equal to 1, we use a more convenient notation for the eigenstates of  $\phi^\dagger(l)\phi(l)$

$$|k\rangle = \phi^\dagger(k)|0_1\rangle \qquad \phi^\dagger(l)\phi(l)|k\rangle = \delta_{lk}|k\rangle .$$

The pilot-state can be decomposed in the basis  $\{|k\rangle\}$ :

$$|\Psi(t)\rangle = \sum_{k=-N}^{k=N-1} \Psi(t, k) |k\rangle .$$

Assume that the beable at the initial time is  $n_l(t_0) = \delta_{lk}$ , corresponding to the state  $|k\rangle = \phi^\dagger(k)|0_1\rangle$ . How does it evolve with time? The hamiltonian matrix element which appears in the transition current (Eq. 14) is

$$\langle l| -iH|k\rangle = \langle 0_1| -i\phi(l)H\phi^\dagger(k)|0_1\rangle = \frac{\delta_{l(k+1)}}{2\delta} - \frac{\delta_{l(k-1)}}{2\delta} .$$

Thus they are only two transitions which are allowed:

$$\begin{aligned} J_{(k+1)k} &= \frac{1}{\delta} \Re[\Psi^*(t, k+1)\Psi(t, k)] \\ J_{(k-1)k} &= -\frac{1}{\delta} \Re[\Psi^*(t, k-1)\Psi(t, k)] \end{aligned} .$$

The quantum can only jump to the first neighbor site, either to the right or the left. Using the relation

$$\Psi^*(t, k-1) = \Psi^*(t, k+1) - 2\delta\nabla\Psi^*(t, k)$$

and keeping only leading terms, we get:

$$\begin{aligned} J_{(k+1)k} &= \frac{1}{\delta} \Re[\Psi^*(t, k+1)\Psi(t, k)] \\ J_{(k-1)k} &= -\frac{1}{\delta} \Re[\Psi^*(t, k+1)\Psi(t, k)] . \end{aligned}$$

These two currents have opposite signs; then there is only one transition. The theory is thus deterministic. Let's consider the case  $\Re[\Psi^*(t, k+1)\Psi(t, k)] \geq 0$ . Then the quantum moves towards the right and its velocity is

$$v = \frac{\delta T_{(k+1)k} dt}{dt} = \frac{\delta J_{(k+1)k}}{|\Psi(t, k)|^2} = \frac{\Re[\Psi^*(t, k+1)\Psi(t, k)]}{|\Psi(t, k)|^2} \quad (44)$$

Now what about the continuum limit? Remember that even sites correspond to upper components of the spinor, whereas odd sites correspond to lower components of the spinor. In the continuum limit, the couple  $(\phi^\dagger(n), \phi^\dagger(n+1))$ , for  $n$  even, tends to the spinor  $(\psi_1^\dagger(x), \psi_2^\dagger(x))$ . There are two sites that converge to a given  $x$ . If  $k$  was even, we still have to consider its companion  $k+1$ . Then let's assume that the beable is  $\delta_{l(k+1)}$  at time  $t_0$ . One can find that the currents are

$$\begin{aligned} J_{(k+2)k+1} &= -\frac{1}{\delta} \Re[\Psi^*(t, k)\Psi(t, k+1)] \\ J_{(k)k+1} &= \frac{1}{\delta} \Re[\Psi^*(t, k)\Psi(t, k+1)] . \end{aligned}$$



We still consider the case  $\Re[\Psi^*(t, k+1)\Psi(t, k)] \geq 0$ . Then only the current  $J_{(k+2)k+1}$  survives and we find that

$$J_{(k+1)k} = J_{(k+2)k+1} = \frac{1}{\delta} \Re[\Psi^*(t, k+1)\Psi(t, k)] .$$

The continuum current is just the continuum version of the sum of the two lattice currents (the one for  $k$  plus the other one for  $k+1$ ):

$$\begin{aligned} J(x, t) &= 2\Re[\langle \Psi(t) | \psi_2^\dagger(t, x) | 0_1 \rangle \langle 0_1 | \psi_1(t, x) | \Psi(t) \rangle] \\ &= \langle \Psi(t) | \psi^\dagger(t, x) | 0_1 \rangle \alpha \langle 0_1 | \psi(t, x) | \Psi(t) \rangle . \end{aligned}$$

## B. Two localized quanta

Now let's consider the case of two localized quanta:

$$\sum_l \phi^\dagger(l) \phi(l) | \Psi(t) \rangle = 2 | \Psi(t) \rangle .$$

We use the same notations: an eigenstate of  $\phi^\dagger(l)\phi(l)$  with eigenvalue  $n_l = \delta_{lk_1} + \delta_{lk_2}$  is denoted

$$|k_1, k_2\rangle = \phi^\dagger(k_1)\phi^\dagger(k_2)|0_1\rangle ,$$

with  $k_1 < k_2$  and  $k_1, k_2 \in \{-N, -N+1, \dots, N-1\}$ , corresponding to the state  $\phi^\dagger(k_1)\phi^\dagger(k_2)|0_1\rangle$ . The most general beable at the initial time is thus  $n_l(t_0) = \delta_{lk_1} + \delta_{lk_2}$ . The hamiltonian matrix element which appears in the transition current is

$$\langle l_1, l_2 | -iH | k_1, k_2 \rangle = - \sum_l \langle 0_1 | \phi(l_2)\phi(l_1)\phi^\dagger(l) \frac{\phi(l+1) - \phi(l-1)}{2\delta} \phi^\dagger(k_1)\phi^\dagger(k_2) | 0_1 \rangle .$$

Using the canonical anti-commutation relations and the fact that the positronic sea is annihilated by any  $\phi(n)$ , and taking into account the constraints  $k_1 < k_2$  and  $l_1 < l_2$ , one finds that

$$\langle l_1, l_2 | -iH | k_1, k_2 \rangle = \frac{\delta_{l_2(k_2+1)}\delta_{l_1k_1} + \delta_{l_1(k_1+1)}\delta_{l_2k_2} - \delta_{l_2(k_2-1)}\delta_{l_1k_1} - \delta_{l_1(k_1-1)}\delta_{l_2k_2}}{2\delta} ,$$

only if  $k_2 \neq k_1 + 1$ , otherwise

$$\langle l_1, l_2 | -iH | k_1, k_2 \rangle = \frac{\delta_{l_2(k_2+1)}\delta_{l_1k_1} - \delta_{l_1(k_1-1)}\delta_{l_2k_2}}{2\delta} .$$

The pilot-state can be decomposed in the basis  $\{\phi^\dagger(n_1)\phi^\dagger(n_2)|0_1\rangle\}$ :

$$|\Psi(t)\rangle = \sum_{n_1} \sum_{n_2 > n_1} \Psi(n_1, n_2, t) |n_1, n_2\rangle .$$

The transition currents are thus

$$\begin{aligned} J_{(k_1, k_2) \rightarrow (k_1-1, k_2)} &= -\frac{1}{\delta} \Re[\Psi^*(k_1-1, k_2) \Psi(k_1, k_2)] \\ J_{(k_1, k_2) \rightarrow (k_1+1, k_2)} &= \frac{1}{\delta} \Re[\Psi^*(k_1+1, k_2) \Psi(k_1, k_2)] \\ J_{(k_1, k_2) \rightarrow (k_1, k_2-1)} &= -\frac{1}{\delta} \Re[\Psi^*(k_1, k_2-1) \Psi(k_1, k_2)] \\ J_{(k_1, k_2) \rightarrow (k_1, k_2+1)} &= \frac{1}{\delta} \Re[\Psi^*(k_1, k_2+1) \Psi(k_1, k_2)] \end{aligned} ,$$

whether  $k_2 = k_1 + 1$  or not, since  $\Psi(n_1, n_1) = 0$ . Keeping only leading terms, we get

$$\begin{aligned} J_{(k_1, k_2) \rightarrow (k_1-1, k_2)} &= -\frac{1}{\delta} \Re[\Psi^*(k_1+1, k_2) \Psi(k_1, k_2)] = -J_{(k_1, k_2) \rightarrow (k_1+1, k_2)} \\ J_{(k_1, k_2) \rightarrow (k_1, k_2-1)} &= -\frac{1}{\delta} \Re[\Psi^*(k_1, k_2+1) \Psi(k_1, k_2)] = -J_{(k_1, k_2) \rightarrow (k_1, k_2+1)} \end{aligned} .$$

Thus there are only two transitions allowed, since the four transitions can be arranged by pairs whose currents have opposite signs. If  $k_1$  and  $k_2$  are even and correspond to  $x_1$  and  $x_2$  in the continuum limit, the two transition currents are the two discrete components of the current

$$\vec{J}_{ee}(x_1, x_2, t) = \begin{pmatrix} \Re[\langle \Psi(t) | \psi_2^\dagger(x_1) \psi_1^\dagger(x_2) | 0_1 \rangle \langle 0_1 | \psi_1(x_2) \psi_1(x_1) | \Psi(t) \rangle] \\ \Re[\langle \Psi(t) | \psi_1^\dagger(x_1) \psi_2^\dagger(x_2) | 0_1 \rangle \langle 0_1 | \psi_1(x_2) \psi_1(x_1) | \Psi(t) \rangle] \end{pmatrix} .$$

Again, it is thus deterministic, since the cases odd-odd and even-odd are similar. There are four couples that correspond to the same  $(x_1, x_2)$  ( $(k_1, k_2)$ ), that we have already considered,  $(k_1 + 1, k_2)$ ,  $((k_1, k_2 + 1)$ , and  $((k_1 + 1, k_2 + 1))$ . Summing their currents, we get

$$\vec{J}(x_1, x_2, t) = \begin{pmatrix} \sum_a \langle \Psi(t) | \psi^\dagger(x_1) \psi_a^\dagger(x_2) | 0_1 \rangle \alpha \langle 0_1 | \psi_a(x_2) \psi(x_1) | \Psi(t) \rangle \\ \sum_a \langle \Psi(t) | \psi_a^\dagger(x_1) \psi^\dagger(x_2) | 0_1 \rangle \alpha \langle 0_1 | \psi(x_2) \psi_a(x_1) | \Psi(t) \rangle \end{pmatrix} .$$

There is also another way to prove that the model is deterministic in the continuum limit. We take the continuum limit and then make a rotation in the configuration space of dimension 2, in order to align the axis  $X_1$  along a preferred direction, for example that of  $\vec{J}_{ee}(x_1, x_2, t)$ , if we consider that case. Then it can be shown that there are two transitions allowed, namely the two transitions along  $\vec{J}_{ee}(x_1, x_2, t)$ . The two currents have equal magnitude but opposite signs, so only one transition remains. Then the particle moves in a deterministic way and the scheme can be applied again for time  $t + dt$ , and so on.

### C. Generalization

Suppose we have  $\omega$  quanta

$$\sum_l \phi^\dagger(l) \phi(l) |\Psi(t)\rangle = \omega |\Psi(t)\rangle .$$

and an initial beable  $n_l(t_0) = \delta_{lk_1} + \delta_{lk_2} + \dots + \delta_{lk_\omega}$ . In the continuum limit, if  $x_j$  is the coordinate corresponding to  $k_j$ , we expect to get the following current for the  $j$ -th coordinate:

$$J_j(t) = \sum_s \sum_{s_1} \dots \sum_{s_\omega} \Psi_{s_1 \dots s_j \dots s_\omega}^*(t, x_1, \dots, x_\omega) \alpha_{s_j s} \Psi_{s_1 \dots s_j \dots s_\omega}(t, x_1, \dots, x_\omega) ,$$

where

$$\Psi_{s_1 \dots s_\omega}(t, x_1, \dots, x_\omega) = \langle 0_1 | \psi_{s_\omega}(x_\omega) \dots \psi_{s_1}(x_1) | \Psi(t) \rangle .$$

It is just a refinement of the two-quanta case. In fact, there will be  $2\omega$  transitions allowed (to first neighbor sites). Those transitions can be arranged by pairs. In each pair, the two currents have equal magnitude but opposite signs. Thus there remains only  $\omega$  transitions (see Eq (14)). Those  $\omega$  currents are just the discrete components of a continuum current in a configuration space of dimension  $\omega$ . The deterministic character can also be proved by making a rotation in the configuration space of dimension  $\omega$ , as it was explained for the two quanta case.

### D. The continuum limit right from the start

We have a physical state  $|\Psi(t)\rangle$  which evolves according to the Schrödinger equation and we know that there is an integer  $\omega$  such that

$$\int d^3 \vec{x} \psi^\dagger(\vec{x}) \psi(\vec{x}) |\Psi(t)\rangle = \omega |\Psi(t)\rangle .$$

Thus  $|\Psi(t)\rangle$  can be decomposed along eigenstates of the fermion number density with fermion number equal to  $\omega$ ; those eigenstates are

$$\{ \psi_{s_1}^\dagger(\vec{x}_1) \dots \psi_{s_\omega}^\dagger(\vec{x}_\omega) | 0_1 \rangle \quad \vec{x}_1, \dots, \vec{x}_\omega \in R^3 \quad s_1, \dots, s_\omega \in \{1, 2, 3, 4\} \} .$$

Thus

$$|\Psi(t)\rangle = \frac{1}{\omega!} \sum_{s_1=1}^{s_1=4} \dots \sum_{s_\omega=1}^{s_\omega=4} \int d^3 \vec{x}_1 \dots d^3 \vec{x}_\omega \Psi_{s_1 \dots s_\omega}(t, \vec{x}_1, \dots, \vec{x}_\omega) \psi_{s_1}^\dagger(\vec{x}_1) \dots \psi_{s_\omega}^\dagger(\vec{x}_\omega) | 0_1 \rangle ,$$

where the wave function is antisymmetric. The universe at time  $t$  is described by a point in a configuration space of dimension  $3\omega$  and by a pilot-state  $|\Psi(t)\rangle$ . In the standard interpretation, the probability density to observe the universe in a configuration  $(\vec{x}_1, \dots, \vec{x}_\omega)$  is

$$\rho_t(\vec{x}_1, \dots, \vec{x}_\omega) = \sum_{s_1=1}^{s_1=4} \dots \sum_{s_\omega=1}^{s_\omega=4} |\langle \Psi(t) | \psi_{s_1}^\dagger(\vec{x}_1) \dots \psi_{s_\omega}^\dagger(\vec{x}_\omega) | 0_1 \rangle|^2 ,$$

and we have

$$\int d^3\vec{x}_1 \dots d^3\vec{x}_\omega \rho_t(\vec{x}_1, \dots, \vec{x}_\omega) = 1 .$$

The time derivative gives a probability density current in the configuration space:

$$\begin{aligned} \frac{d}{dt} \int d^3\vec{x}_1 \dots d^3\vec{x}_\omega \rho_t(\vec{x}_1, \dots, \vec{x}_\omega) &= \sum_{s_1=1}^{s_1=4} \dots \sum_{s_\omega=1}^{s_\omega=4} \int d^3\vec{x}_1 \dots d^3\vec{x}_\omega \\ \frac{d}{dt} [\langle \Psi(t_0) | \psi_{s_1}^\dagger(\vec{x}_1, t) \dots \psi_{s_\omega}^\dagger(\vec{x}_\omega, t) | 0_1 \rangle \langle 0_1 | \psi_{s_\omega}(\vec{x}_\omega, t) \dots \psi_{s_1}(\vec{x}_1, t) | \Psi(t_0) \rangle] &= 0 \end{aligned} ,$$

where we have switched to the Heisenberg picture. It can be simplified, knowing that

$$i \frac{d\psi(t, x)}{dt} = -i\vec{\alpha} \cdot \nabla \psi(t, x) + m\beta \psi(t, x) .$$

Terms containing  $\beta$  cannot contribute. Thus we obtain the following current for the  $j$ -th coordinate:

$$\begin{aligned} \vec{J}_j(\vec{x}_1, \dots, \vec{x}_\omega, t) &= \sum_{s=1}^{s=4} \sum_{s_1=1}^{s_1=4} \dots \sum_{s_\omega=1}^{s_\omega=4} \langle \Psi(t_0) | \psi_{s_1}^\dagger(\vec{x}_1, t) \dots \psi_{s_j}^\dagger(\vec{x}_j, t) \dots \psi_{s_\omega}^\dagger(\vec{x}_\omega, t) | 0_1 \rangle \\ &\quad \alpha_{s_j s} \langle 0_1 | \psi_{s_\omega}(\vec{x}_\omega, t) \dots \psi_s(\vec{x}_j, t) \dots \psi_{s_1}(\vec{x}_1, t) | \Psi(t_0) \rangle . \end{aligned}$$

Define  $\vec{A} = (\vec{a}_1, \dots, \vec{a}_\omega)$ ,  $\forall \vec{a} \in R^3$ . If  $\vec{X}(t)$  is the position of the universe in the configuration space, at time  $t$ , its velocity is thus

$$\left. \frac{\vec{J}(\vec{X}, t)}{\rho_t(\vec{X})} \right|_{\vec{X}=\vec{X}(t)} .$$

## VI. QUANTUM NON-LOCALITY

The Bell model is non-local, but this is a necessary property of any realistic interpretation of quantum field theory, following the EPR paradox, Bell's inequality and experiments. We just want to show it explicitly. Consider the case of two negative charges moving in the

positronic sea. Then there is an interaction among these two charges, by the Pauli Principle; the wave function  $\Psi_{s_1 s_2}(t, x_1, x_2)$  is antisymmetric. The least entangled state, satisfying the antisymmetry requirement, is

$$\Psi_{s_1 s_2}(t, x_1, x_2) = \chi_{s_1}(t, x_1)\Phi_{s_2}(t, x_2) - \Phi_{s_1}(t, x_1)\chi_{s_2}(t, x_2) . \quad (45)$$

The currents are

$$\begin{aligned} J_1(t, x_1, x_2) &= \sum_{s_2} [\Psi_{1s_2}^*(t, x_1, x_2)\Psi_{2s_2}(t, x_1, x_2) + \Psi_{2s_2}^*(t, x_1, x_2)\Psi_{1s_2}(t, x_1, x_2)] , \\ J_2(t, x_1, x_2) &= \sum_{s_1} [\Psi_{s_1 1}^*(t, x_1, x_2)\Psi_{s_1 2}(t, x_1, x_2) + \Psi_{s_1 2}^*(t, x_1, x_2)\Psi_{s_1 1}(t, x_1, x_2)] . \end{aligned}$$

Substituting  $\Psi_{s_1 s_2}(t, x_1, x_2)$  by the right-hand part of Eq. (45), we get, for  $J_1$ :

$$\begin{aligned} J_1(t, x_1, x_2) &= (\chi_1^* \chi_2 + \chi_2^* \chi_1)(t, x_1)(|\Phi_1|^2 + |\Phi_2|^2)(t, x_2) \\ &\quad + (\Phi_1^* \Phi_2 + \Phi_2^* \Phi_1)(t, x_1)(|\chi_1|^2 + |\chi_2|^2)(t, x_2) \\ &\quad - (\chi_1^* \Phi_2 + \chi_2^* \Phi_1)(t, x_1)(\Phi_1^* \chi_1 + \Phi_2^* \chi_2)(t, x_2) \\ &\quad - (\Phi_1^* \chi_2 + \Phi_2^* \chi_1)(t, x_1)(\chi_1^* \Phi_1 + \chi_2^* \Phi_2)(t, x_2) . \end{aligned}$$

This general form is inconsistent with the existence of two real currents  $J_1^A$  and  $J_1^B$  such that

$$J_1(t, x_1, x_2) = J_1^A(t, x_1)J_1^B(t, x_2) ,$$

thus the model is clearly non-local: the velocity of one of the charges, at time  $t$ , depends on its position, on the pilot-state, as well as on the position of the other negative charge, at the same time.

## VII. QUANTUM FIELD THEORY IN A FIXED SECTOR OF THE FOCK SPACE

It is interesting to note that quantum field theory calculations can be done in the same way as they are made in non-relativistic quantum mechanics, at least if we only consider fermions. Let's consider the following model, fermions interacting through a quartic term:

$$H = \int d^3 \vec{x} (\psi^\dagger(\vec{x})[-i\vec{\alpha} \cdot \nabla + m\beta]\psi(\vec{x}) + g(\psi^\dagger(\vec{x})\beta\psi(\vec{x}))^2) .$$

Assume that there are two negative charges in the positronic sea:

$$\int d^3 \vec{x} \psi^\dagger(\vec{x})\psi(\vec{x})|\Psi(t)\rangle = 2|\Psi(t)\rangle ,$$

with

$$i \frac{d|\Psi(t)\rangle}{dt} = H|\Psi(t)\rangle .$$

Then the pilot-state can be decomposed along eigenstates of the fermion number density, with fermion number equal to two:

$$|\Psi(t)\rangle = \frac{1}{2!} \sum_{s_1=1}^{s_1=4} \sum_{s_2=1}^{s_2=4} \int d^3\vec{x}_1 d^3\vec{x}_2 \Psi_{s_1 s_2}(t, \vec{x}_1, \vec{x}_2) \psi_{s_1}^\dagger(\vec{x}_1) \psi_{s_2}^\dagger(\vec{x}_2) |0_1\rangle .$$

Substituting the pilot-state by the right-hand part of the previous equation in the Schrödinger equation, and projecting onto a state  $\psi_{s_1}^\dagger(\vec{x}_1) \psi_{s_2}^\dagger(\vec{x}_2) |0_1\rangle$ , we get

$$\begin{aligned} i \frac{d}{dt} \Psi_{s_1 s_2}(t, \vec{x}_1, \vec{x}_2) = & (\beta \Psi(\vec{x}_1, \vec{x}_2))_{s_1 s_2} - (\beta \Psi(\vec{x}_2, \vec{x}_1))_{s_2 s_1} \\ & - i(\alpha \cdot \nabla_{\vec{x}_1} \Psi_t(\vec{x}_1, \vec{x}_2))_{s_1 s_2} + i(\alpha \cdot \nabla_{\vec{x}_2} \Psi_t(\vec{x}_2, \vec{x}_1))_{s_2 s_1} \\ & + (\beta \Psi(t, \vec{x}_1, \vec{x}_2) \beta^T)_{s_1 s_2} \delta(\vec{x}_1 - \vec{x}_2) - (\beta \Psi(t, \vec{x}_2, \vec{x}_1) \beta^T)_{s_2 s_1} \delta(\vec{x}_1 - \vec{x}_2) \quad , \end{aligned}$$

where we have dropped an infinite constant.

## VIII. CONCLUSION

We have obtained the continuum limit of the Bell model, for fermions living in a one-dimensional space, using a staggered lattice and we have also shown that we could build the continuum Bell model directly. Physically, it is a theory of negative charges moving in a positronic sea. There is an underlying assumption about the state of the universe, namely that it is an eigenstate of the fermion number (which is always true), with a finite eigenvalue. That follows naturally from the Bell model itself. Can one build a similar interpretation for the Klein-Gordon quantum field theory? It seems that the answer is no, for it is impossible to define a state annihilated by a charge creator in the Klein-Gordon quantum field theory. Another point worth mentioning is that the construction of the Bell model has nothing to do with the equation of motion being linear. We could use a Van der Waerden field and obtain the same results. Only the Pauli exclusion principle is at work.

## Acknowledgments

The author would like to thank Jean Bricmont and Thomas Durt, for taking the time to discuss the main ideas with him.

## APPENDIX A: COMMUTATOR $[N, \psi^\dagger(\vec{x})\psi(\vec{x})]$

We want to show that

$$[N, \psi^\dagger(\vec{x})\psi(\vec{x})] \neq 0 , \quad (\text{A1})$$

with

$$N = \sum_r \int d^3\vec{k} [c_r^\dagger(\vec{k})c_r(\vec{k}) + d_r^\dagger(\vec{k})d_r(\vec{k})] . \quad (\text{A2})$$

We use the following relation ( $F$  stands for fermion):

$$[F_1 F_2, F_3 F_4] = F_1 [F_2, F_3 F_4] + [F_1, F_3 F_4] F_2 \quad (\text{A3})$$

$$= F_1 \{F_2, F_3\} F_4 - F_1 F_3 \{F_2, F_4\} + \{F_1, F_3\} F_4 F_2 - F_3 \{F_1, F_4\} F_2 . \quad (\text{A4})$$

Let's recall the expressions of the spinor fields:

$$\psi(\vec{x}) = \sqrt{\frac{1}{(2\pi)^3}} \sum_s \int d^3\vec{p} \sqrt{\frac{m}{E_{\vec{p}}}} [u_s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} c_s(\vec{p}) + v_s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} d_s^\dagger(\vec{p})] \quad (\text{A5})$$

$$\psi^\dagger(\vec{x}) = \sqrt{\frac{1}{(2\pi)^3}} \sum_s \int d^3\vec{p} \sqrt{\frac{m}{E_{\vec{p}}}} [u_s^\dagger(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} c_s^\dagger(\vec{p}) + v_s^\dagger(\vec{p}) e^{i\vec{p}\cdot\vec{x}} d_s(\vec{p})] . \quad (\text{A6})$$

By using the anti-commutation relations

$$\{c_s(\vec{k}), c_r^\dagger(\vec{p})\} = \delta_{sr} \delta^3(\vec{k} - \vec{p}) \quad \{d_s(\vec{k}), d_r^\dagger(\vec{p})\} = \delta_{sr} \delta^3(\vec{k} - \vec{p}) , \quad (\text{A7})$$

and all other anti-commutators vanishing, we find that

$$\{\psi_a^\dagger(\vec{x}), c_r(\vec{k})\} = \sqrt{\frac{1}{(2\pi)^3}} \sqrt{\frac{m}{E_{\vec{k}}}} u_{ar}^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \quad \{\psi_a(\vec{x}), c_r(\vec{k})\} = 0 \quad (\text{A8})$$

$$\{\psi_a(\vec{x}), c_r^\dagger(\vec{k})\} = \sqrt{\frac{1}{(2\pi)^3}} \sqrt{\frac{m}{E_{\vec{k}}}} u_{ar}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \quad \{\psi_a^\dagger(\vec{x}), c_r^\dagger(\vec{k})\} = 0 \quad (\text{A9})$$

$$\{\psi_a(\vec{x}), d_r(\vec{k})\} = \sqrt{\frac{1}{(2\pi)^3}} \sqrt{\frac{m}{E_{\vec{k}}}} v_{ar}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \quad \{\psi_a^\dagger(\vec{x}), d_r(\vec{k})\} = 0 \quad (\text{A10})$$

$$\{\psi_a^\dagger(\vec{x}), d_r^\dagger(\vec{k})\} = \sqrt{\frac{1}{(2\pi)^3}} \sqrt{\frac{m}{E_{\vec{k}}}} v_{ar}^\dagger(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \quad \{\psi_a(\vec{x}), d_r^\dagger(\vec{k})\} = 0 , \quad (\text{A11})$$

so that

$$[\psi_a^\dagger(\vec{x})\psi_a(\vec{x}), \sum_r \int d^3\vec{k} c_r^\dagger(\vec{k})c_r(\vec{k})] = \quad (\text{A12})$$

$$\sum_r \int d^3\vec{k} (\psi_a^\dagger(\vec{x})\{\psi_a(\vec{x}), c_r^\dagger(\vec{k})\}c_r(\vec{k}) - c_r^\dagger(\vec{k})\{\psi_a^\dagger(\vec{x}), c_r(\vec{k})\}\psi_a(\vec{x})) = \quad (\text{A13})$$

$$\frac{m^2}{(2\pi)^3} \sum_{s,r} \int \frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}} [u_s^\dagger(\vec{p})u_r(\vec{k})e^{-i(\vec{p}-\vec{k})\cdot\vec{x}}c_s^\dagger(\vec{p})c_r(\vec{k}) + v_s^\dagger(\vec{p})u_r(\vec{k})e^{i(\vec{p}+\vec{k})\cdot\vec{x}}d_s(\vec{p})c_r(\vec{k})] - \quad (\text{A14})$$

$$\frac{m^2}{(2\pi)^3} \sum_{s,r} \int \frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}} [u_r^\dagger(\vec{k})u_s(\vec{p})e^{i(\vec{p}-\vec{k})\cdot\vec{x}}c_r^\dagger(\vec{k})c_s(\vec{p}) + u_r^\dagger(\vec{k})v_s(\vec{p})e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}c_r^\dagger(\vec{k})d_s^\dagger(\vec{p})] \quad . \quad (\text{A15})$$

Since  $r$ ,  $s$ ,  $\vec{p}$  and  $\vec{k}$  are dummy variables, we find that

$$[\psi_a^\dagger(\vec{x})\psi_a(\vec{x}), \sum_r \int d^3\vec{k} c_r^\dagger(\vec{k})c_r(\vec{k})] = \quad (\text{A16})$$

$$\frac{m^2}{(2\pi)^3} \sum_{s,r} \int \frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}} [v_s^\dagger(\vec{p})u_r(\vec{k})e^{i(\vec{p}+\vec{k})\cdot\vec{x}}d_s(\vec{p})c_r(\vec{k})] - \quad (\text{A17})$$

$$\frac{m^2}{(2\pi)^3} \sum_{s,r} \int \frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}} [u_r^\dagger(\vec{k})u_s(\vec{p})e^{i(\vec{p}-\vec{k})\cdot\vec{x}}c_r^\dagger(\vec{k})c_s(\vec{p})] \quad . \quad (\text{A18})$$

In the same way, we obtain

$$[\psi_a^\dagger(\vec{x})\psi_a(\vec{x}), \sum_r \int d^3\vec{k} d_r^\dagger(\vec{k})d_r(\vec{k})] = \quad (\text{A19})$$

$$\sum_r \int d^3\vec{k} (-\psi_a^\dagger(\vec{x})d_r^\dagger(\vec{k})\{\psi_a(\vec{x}), d_r(\vec{k})\} + \{\psi_a^\dagger(\vec{x}), d_r^\dagger(\vec{k})\}d_r(\vec{k})\psi_a(\vec{x})) = - \quad (\text{A20})$$

$$\frac{m^2}{(2\pi)^3} \sum_{s,r} \int \frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}} [u_s^\dagger(\vec{p})v_r(\vec{k})e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}c_s^\dagger(\vec{p})d_r^\dagger(\vec{k}) + v_s^\dagger(\vec{p})v_r(\vec{k})e^{i(\vec{p}-\vec{k})\cdot\vec{x}}d_s(\vec{p})d_r^\dagger(\vec{k})] + \quad (\text{A21})$$

$$\frac{m^2}{(2\pi)^3} \sum_{s,r} \int \frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}} [v_r^\dagger(\vec{k})u_s(\vec{p})e^{i(\vec{p}+\vec{k})\cdot\vec{x}}d_r(\vec{k})c_s(\vec{p}) + v_r^\dagger(\vec{k})v_s(\vec{p})e^{-i(\vec{p}-\vec{k})\cdot\vec{x}}d_r(\vec{k})d_s^\dagger(\vec{p})] \quad . \quad (\text{A22})$$



This can be simplified to

$$[\psi_a^\dagger(\vec{x})\psi_a(\vec{x}), \sum_r \int d^3\vec{k} d_r^\dagger(\vec{k})d_r(\vec{k})] = \quad (\text{A23})$$

$$\frac{m^2}{(2\pi)^3} \sum_{s,r} \int \frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}} [u_s^\dagger(\vec{p})v_r(\vec{k})e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}c_s^\dagger(\vec{p})d_r^\dagger(\vec{k})] + \quad (\text{A24})$$

$$\frac{m^2}{(2\pi)^3} \sum_{s,r} \int \frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}} [v_r^\dagger(\vec{k})u_s(\vec{p})e^{i(\vec{p}+\vec{k})\cdot\vec{x}}d_r(\vec{k})c_s(\vec{p})] \quad . \quad (\text{A25})$$

Putting the two results together, we get

$$[\psi_a^\dagger(\vec{x})\psi_a(\vec{x}), N] = \quad (\text{A26})$$

$$\frac{2m^2}{(2\pi)^3} \sum_{s,r} \int \frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}} [u_s^\dagger(\vec{p})v_r(\vec{k})e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}c_s^\dagger(\vec{p})d_r^\dagger(\vec{k})] + \quad (\text{A27})$$

$$\frac{2m^2}{(2\pi)^3} \sum_{s,r} \int \frac{d^3\vec{p}d^3\vec{k}}{\sqrt{E_{\vec{p}}E_{\vec{k}}}} [v_r^\dagger(\vec{k})u_s(\vec{p})e^{i(\vec{p}+\vec{k})\cdot\vec{x}}d_r(\vec{k})c_s(\vec{p})] \quad , \quad (\text{A28})$$

which is not equal to zero, even if we think about fields as distributions. If we start from the state  $d_s^\dagger(p_0)c_s^\dagger(p_0)|0\rangle$ , it is clear that there are well-behaved functions  $f$  such that

$$\langle 0 | \int d^3\vec{x} f(\vec{x}) [\psi^\dagger(\vec{x})\psi(\vec{x}), N] | d_s^\dagger(p_0)c_s^\dagger(p_0)|0\rangle \neq 0 \quad . \quad (\text{A29})$$

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